A THEORY OF QUANTUM GRAVITY FROM FIRST PRINCIPLES

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Abstract. When quantum fields are studied on manifolds with boundary, the corresponding one-loop quantum theory for bosonic gauge fields with linear covariant gauges needs the assignment of suitable boundary conditions for elliptic differential operators of Laplace type. There are however deep reasons to modify such a scheme and allow for pseudo-differential boundary-value problems. When the boundary operator is allowed to be pseudo-differential while remaining a projector, the conditions on its kernel leading to strong ellipticity of the boundary-value problem are studied in detail. This makes it possible to develop a theory of one-loop quantum gravity from first principles only, i.e. the physical principle of invariance under infinitesimal diffeomorphisms and the mathematical requirement of a strongly elliptic theory.

The space-time approach to quantum mechanics and quantum field theory has led to several deep developments in the understanding of quantum theory and space-time structure at very high energies.[1,2] In particular, we are here concerned with the choice of boundary conditions. On using path integrals, which lead, in principle, to the appropriate formulation of the ideas of Feynman, DeWitt and many other authors,[2−5] the assignment of boundary conditions consists of two main steps:

(i) Choice of Riemannian geometries and field configurations to be included in the path-integral representation of transition amplitudes.

(ii) Choice of boundary data to be imposed on the hypersurfaces Σ_1 and Σ_2 bounding the given space-time region.

The main object of our investigation is the second problem of such a list, when a one-loop approximation is studied for a bosonic gauge theory in linear covariant gauges. The well posed mathematical formulation relies on the "Euclidean approach", i.e., in geometric language, on the use of differentiable manifolds endowed with positive-definite metrics g, so that space-time is actually replaced by an m-dimensional Riemannian space (M, g).

In particular, in Euclidean quantum gravity, mixed boundary conditions on metric perturbations h_{cd} occur naturally if one requires their complete invariance under infinitesimal diffeomorphisms, as is proved in detail in Ref. 6. On denoting by n^a the inward-pointing unit normal to the boundary, by

$$q^a_{\ b} \equiv \delta^a_{\ b} - n^a n_b \tag{1}$$

the projector of tensor fields onto ∂M, with associated projection operator

$$\Pi_{ab}^{\ \ cd} \equiv q^c_{\ (a} \, q^d_{\ b)}, \tag{2}$$

the gauge-invariant boundary conditions for one-loop quantum gravity read[6]

$$\Big[\Pi_{ab}^{\ \ cd} h_{cd}\Big]_{\partial M} = 0, \tag{3}$$

$$\Big[\Phi_a(h)\Big]_{\partial M} = 0, \tag{4}$$

where Φ_a is the gauge-averaging functional necessary to obtain an invertible operator ${P_{ab}}^{cd}$ on metric perturbations. When ${P_{ab}}^{cd}$ is chosen to be of Laplace type, Φ_a reduces to the familiar de Donder term

$$\Phi_a(h) = \nabla^b \Big(h_{ab} - \frac{1}{2} g_{ab} g^{cd} h_{cd} \Big) = E_a^{\ bcd} \nabla_b h_{cd}, \tag{5}$$

where E^{abcd} is the DeWitt supermetric on the vector bundle of symmetric rank-two tensor fields over M (g being the metric on M):

$$E^{abcd} \equiv \frac{1}{2} \Big(g^{ac} g^{bd} + g^{ad} g^{bc} - g^{ab} g^{cd} \Big). \tag{6}$$

The boundary conditions (3) and (4) can then be cast in the Gilkey–Smith form:[7]

$$\begin{pmatrix} \Pi & 0 \\ \Lambda & I - \Pi \end{pmatrix} \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi_{;N}]_{\partial M} \end{pmatrix} = 0. \tag{7}$$

However, the work in Ref. 6 has shown that an operator of Laplace type on metric perturbations is then incompatible with the requirement of strong ellipticity of the boundary-value problem, because the operator Λ contains tangential derivatives of metric perturbations.

To take care of this serious drawback, the work in Ref. 8 has proposed to consider in the boundary condition (4) a gauge-averaging functional given by the de Donder term (5) plus an integro-differential operator on metric perturbations, i.e.

$$\Phi_a(h) \equiv E_a^{\ bcd} \nabla_b h_{cd} + \int_M \zeta_a^{\ cd}(x, x') h_{cd}(x') dV'. \tag{8}$$

We now begin by remarking that the resulting boundary conditions can be cast in the form

$$\begin{pmatrix} \Pi & 0 \\ \Lambda + \widetilde{\Lambda} & I - \Pi \end{pmatrix} \begin{pmatrix} [\varphi]_{\partial M} \\ [\varphi_{;N}]_{\partial M} \end{pmatrix} = 0, \tag{9}$$

where $\widetilde{\Lambda}$ reflects the occurrence of the integral over M in Eq. (8). It is convenient to work first in a general way and then consider the form taken by these operators in the gravitational case. On requiring that the resulting boundary operator

$$\mathcal{B} \equiv \begin{pmatrix} \Pi & 0 \\ \Lambda + \widetilde{\Lambda} & I - \Pi \end{pmatrix} \tag{10}$$

should remain a projector: $\mathcal{B}^2 = \mathcal{B}$, we find the condition

$$(\Lambda + \widetilde{\Lambda})\Pi - \Pi(\Lambda + \widetilde{\Lambda}) = 0, \tag{11}$$

which reduces to

$$\Pi\widetilde{\Lambda} = \widetilde{\Lambda}\Pi, \tag{12}$$

by virtue of the property $\Pi\Lambda = \Lambda\Pi = 0$ considered in Ref. 6.

In Euclidean quantum gravity at one-loop level, Eq. (12) leads to

$$\Pi_{a\ \ c}^{\ b\ \ r}(x) \int_M \zeta_b^{\ cq}(x,x')h_{qr}(x')dV' = \int_M \zeta_a^{\ cd}(x,x')\Pi_{cd}^{\ \ qr}(x')h_{qr}(x')dV', \tag{13}$$

which can be re-expressed in the form

$$\int_M \left[\Pi_{a\ \ c}^{\ b\ \ r}(x)\zeta_b^{\ cq}(x,x') - \zeta_a^{\ cd}(x,x')\Pi_{cd}^{\ \ qr}(x')\right] h_{qr}(x')dV' = 0. \tag{14}$$

Since this should hold for all $h_{qr}(x')$, it eventually leads to the vanishing of the term in square brackets in the integrand. The notation $\zeta_b^{\ cq}(x,x')$ is indeed rather awkward, because there is an even number of arguments, i.e. x and x', with an odd number of indices. Hereafter, we therefore assume that a vector field T and kernel $\widetilde{\zeta}$ exist such that

$$\zeta_b^{\ cq}(x,x') \equiv T^p(x)\widetilde{\zeta}_{bp}^{\ \ cq}(x,x') \equiv T^p\widetilde{\zeta}_{bp}^{\ \ c'q'}. \tag{15}$$

The projector condition (12) is therefore satisfied if and only if[9]

$$T^p(x)\left[\Pi_{a\ \ c}^{\ b\ \ r}(x)\widetilde{\zeta}_{bp}^{\ \ cq}(x,x') - \widetilde{\zeta}_{ap}^{\ \ cd}(x,x')\Pi_{cd}^{\ \ qr}(x')\right] = 0. \tag{16}$$

We are now concerned with the issue of ellipticity of the boundary-value problem of one-loop quantum gravity. For this purpose, we begin by recalling what is known about ellipticity of the Laplacian (hereafter P) on a Riemannian manifold with smooth boundary. This concept is studied in terms of the leading symbol of P. It is indeed well known that the Fourier transform makes it possible to associate to a differential operator of order k a polynomial of degree k, called the characteristic polynomial or symbol. The leading symbol, σ_L, picks out the highest order part of this polynomial. For the Laplacian, it reads

$$\sigma_L(P;x,\xi) = |\xi|^2 I = g^{\mu\nu}\xi_\mu \xi_\nu I. \tag{17}$$

With a standard notation, (x, ξ) are local coordinates for $T^*(M)$, the cotangent bundle of M. The leading symbol of P is trivially elliptic in the interior of M, since the right-hand side of (17) is positive-definite, and one has

$$\det(\sigma_L(P;x,\xi) - \lambda) = (|\xi|^2 - \lambda)^{\dim V} \neq 0, \tag{18}$$

for all $\lambda \in \mathcal{C} - \mathbf{R}_+$. In the presence of a boundary, however, one needs a more careful definition of ellipticity. First, for a manifold M of dimension m, the m coordinates x are split into $m-1$ local coordinates on ∂M, hereafter denoted by $\left\{\hat{x}^k\right\}$, and r, the geodesic distance to the boundary. Moreover, the m coordinates ξ_μ are split into $m-1$ coordinates $\{\zeta_j\}$ (with ζ being a cotangent vector on the boundary), jointly with a real parameter $\omega \in T^*(\mathbf{R})$. At a deeper level, all this reflects the split

$$T^*(M) = T^*(\partial M) \oplus T^*(\mathbf{R}) \tag{19}$$

in a neighbourhood of the boundary.[6,10]

The ellipticity we are interested in requires now that σ_L should be elliptic in the interior of M, as specified before, and that strong ellipticity should hold. This means that a unique solution exists of the differential equation obtained from the leading symbol:

$$\left[\sigma_L\left(P; \left\{\hat{x}^k\right\}, r=0, \{\zeta_j\}, \omega \rightarrow -i\frac{\partial}{\partial r}\right) - \lambda\right]\varphi(r) = 0, \tag{20}$$

subject to the boundary conditions

$$\sigma_g(B)\left(\left\{\hat{x}^k\right\},\{\zeta_j\}\right)\psi(\varphi)=\psi'(\varphi) \tag{21}$$

and to the asymptotic condition

$$\lim_{r\to\infty}\varphi(r)=0. \tag{22}$$

In Eq. (21), σ_g is the *graded leading symbol* of the boundary operator in the local coordinates $\left\{\hat{x}^k\right\},\{\zeta_j\}$, and is given by

$$\sigma_g(B)=\begin{pmatrix}\Pi & 0\\ i\Gamma^j\zeta_j & I-\Pi\end{pmatrix}. \tag{23}$$

Roughly speaking, the above construction uses Fourier transform and the inward geodesic flow to obtain the ordinary differential equation (20) from the Laplacian, with corresponding Fourier transform (21) of the original boundary conditions. The asymptotic condition (22) picks out the solutions of Eq. (20) which satisfy Eq. (21) with arbitrary boundary data $\psi'(\varphi)\in C^\infty(W',\partial M)$ for W' a vector bundle over the boundary, and vanish at infinite geodesic distance to the boundary. When all the above conditions are satisfied $\forall\zeta\in T^*(\partial M),\forall\lambda\in\mathcal{C}-\mathbf{R}_+,\forall(\zeta,\lambda)\neq(0,0)$ and $\forall\psi'(\varphi)\in C^\infty(W',\partial M)$, the boundary-value problem (P,B) for the Laplacian is said to be strongly elliptic with respect to the cone $\mathcal{C}-\mathbf{R}_+$.

However, when the gauge-averaging functional (8) is used in the boundary condition (5), the work in Ref. 8 has proved that the operator on metric perturbations takes the form of an operator of Laplace type ${P_{ab}}^{cd}$ plus an integral operator ${G_{ab}}^{cd}$. Explicitly, one finds[8] (with $R^a_{\ bcd}$ being the Riemann curvature of the background geometry (M,g))

$${P_{ab}}^{cd}={E_{ab}}^{cd}(-\Box+R)-2{E_{ab}}^{qf}R^c_{\ qpf}g^{dp}-{E_{ab}}^{pd}R_p^{\ c}-{E_{ab}}^{cp}R_p^{\ d}, \tag{24}$$

$${G_{ab}}^{cd}={U_{ab}}^{cd}+{V_{ab}}^{cd}, \tag{25}$$

where

$${U_{ab}}^{cd}h_{cd}(x)=-2E_{rsab}\nabla^r\int_M T^p(x)\widetilde{\zeta}^{s\ \ cd}_{\ p}(x,x')h_{cd}(x')dV', \tag{26}$$

$$h^{ab}V_{ab}{}^{cd}h_{cd}(x) = \int_{M^2} h^{ab}(x')T^q(x)\widetilde{\zeta}_{pqab}(x,x')T^r(x)\widetilde{\zeta}^p_{\ r}{}^{cd}(x,x'')h_{cd}(x'')dV'dV''. \tag{27}$$

We now assume that the operator on metric perturbations, which is so far an integro-differential operator defined by a kernel, is also pseudo-differential. This means that it can be characterized by suitable regularity properties obeyed by the symbol. More precisely, let S^d be the set of all symbols $p(x,\xi)$ such that

(1) p is C^∞ in (x,ξ), with compact x support.

(2) For all (α,β), there exist constants $C_{\alpha,\beta}$ for which

$$\left|(-i)^{\sum_{k=1}^m(\alpha_k+\beta_k)}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}...\left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}\left(\frac{\partial}{\partial \xi_1}\right)^{\beta_1}...\left(\frac{\partial}{\partial \xi_m}\right)^{\beta_m}p(x,\xi)\right|$$

$$\leq C_{\alpha,\beta}\left(1+\sqrt{g^{ab}(x)\xi_a\xi_b}\right)^{d-\sum_{k=1}^m\beta_k}, \tag{28}$$

for some *real* (not necessarily positive) value of d. The associated pseudo-differential operator, defined on the Schwarz space and taking values in the set of smooth functions on M with compact support:

$$P : \mathcal{S} \rightarrow C_c^\infty(M)$$

acts according to

$$Pf(x) \equiv \int e^{i(x-y)\cdot\xi}p(x,\xi)f(y)\mu(y,\xi), \tag{29}$$

where $\mu(y,\xi)$ is here meant to be the invariant integration measure with respect to $y_1,...,y_m$ and $\xi_1,...,\xi_m$. Actually, one first gives the definition for pseudo-differential operators $P : \mathcal{S} \rightarrow C_c^\infty(\mathbf{R}^m)$, eventually proving that a coordinate-free definition can be given and extended to smooth Riemannian manifolds.[10]

In the presence of pseudo-differential operators, both ellipticity in the interior of M and strong ellipticity of the boundary-value problem need a more involved formulation. In our paper, inspired by the flat-space analysis in Ref. 11, we make the following requirements.[9]

(i) Ellipticity in the Interior

Let U be an open subset with compact closure in M, and consider an open subset U_1 whose closure $\overline{U}_1$ is properly included into U: $\overline{U}_1 \subset U$. If p is a symbol of order d on U, it is said to be *elliptic* on U_1 if there exists an open set U_2 which contains $\overline{U}_1$ and positive constants C_0, C_1 so that

$$|p(x,\xi)|^{-1} \leq C_1(1+|\xi|)^{-d}, \tag{30}$$

for $|\xi| \geq C_0$ and $x \in U_2$, where $|\xi| \equiv \sqrt{g^{ab}(x)\xi_a \xi_b}$. The corresponding operator P is then elliptic.

(ii) Strong Ellipticity in the Absence of Boundaries

Let us assume that the symbol under consideration is *polyhomogeneous*, in that it admits an asymptotic expansion of the form

$$p(x,\xi) \sim \sum_{l=0}^{\infty} p_{d-l}(x,\xi), \tag{31}$$

where each term p_{d-l} has the *homogeneity property*

$$p_{d-l}(x,t\xi) = t^{d-l} p_{d-l}(x,\xi) \;\; \text{if} \;\; t \geq 1 \;\; \text{and} \;\; |\xi| \geq 1. \tag{32}$$

The leading symbol is then, by definition,

$$p^0(x,\xi) \equiv p_d(x,\xi). \tag{33}$$

Strong ellipticity in the absence of boundaries is formulated in terms of the leading symbol, and it requires that

$$\mathrm{Re}\, p^0(x,\xi) \geq c(x)|\xi|^d, \tag{34}$$

where $x \in M$ and $|\xi| \geq 1$, c being a positive function on M. It can then be proved that the Gärding inequality holds, according to which, for any $\varepsilon > 0$,

$$\mathrm{Re}(Pu,u) \geq b\|u\|^2_{\frac{d}{2}} - b_1\|u\|^2_{\frac{d}{2}-\varepsilon} \;\; \text{for} \;\; u \in H^{\frac{d}{2}}(M), \tag{35}$$

with $b > 0$.

(iii) Strong Ellipticity in the Presence of Boundaries

The homogeneity property (32) only holds for $t \geq 1$ and $|\xi| \geq 1$. Consider now the case $l = 0$, for which one obtains the leading symbol which plays the key role in the definition of ellipticity. If $p^0(x,\xi) \equiv p_d(x,\xi) \equiv \sigma_L(P;x,\xi)$ is not a polynomial (which corresponds to the genuinely pseudo-differential case) while being a homogeneous function of ξ, it is irregular at $\xi = 0$. When $|\xi| \leq 1$, the only control over the leading symbol is provided by estimates of the form[11]

$$\left|(-i)^{\sum_{k=1}^{m}(\alpha_k+\beta_k)}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}\left(\frac{\partial}{\partial \xi_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial \xi_m}\right)^{\beta_m} p^0(x,\xi)\right|$$
$$\leq c(x)\langle\xi\rangle^{d-|\alpha|}. \tag{36}$$

We therefore come to appreciate the problematic aspect of symbols of pseudo-differential operators.[11] The singularity at $\xi = 0$ can be dealt with either by modifying the leading symbol for small ξ to be a C^∞ function (at the price of loosing the homogeneity there), or by keeping the strict homogeneity and dealing with the singularity at $\xi = 0$.[11]

On the other hand, we are interested in a definition of strong ellipticity of pseudo-differential boundary-value problems that reduces to Eqs. (20)–(22) when both P and the boundary operator reduce to the form considered in Ref. 6. For this purpose, and bearing in mind the occurrence of singularities in the leading symbols of P and of the boundary operator, we make the following requirements.[9]

Let $(P+G)$ be a pseudo-differential operator subject to boundary conditions described by the pseudo-differential boundary operator $\mathcal{B}$ (the consideration of $(P+G)$ rather than only P is necessary to achieve self-adjointness, as is described in detail in Refs. 11 and 12). The pseudo-differential boundary-value problem $((P+G),\mathcal{B})$ is strongly elliptic with respect to $\mathcal{C} - \mathbf{R}_+$ if:

(I) The inequalities (30) and (34) hold;

(II) There exists a unique solution of the equation

$$\left[\sigma_L\left((P+G);\left\{\hat{x}^k\right\},r=0,\left\{\zeta_j\right\},\omega\to -i\frac{\partial}{\partial r}\right)-\lambda\right]\varphi(r)=0, \tag{20$'$}$$

subject to the boundary conditions

$$\sigma_L(\mathcal{B})\left(\left\{\hat{x}^k\right\},\left\{\zeta_j\right\}\right)\psi(\varphi)=\psi'(\varphi) \tag{21$'$}$$

and to the asymptotic condition (22). It should be stressed that, unlike the case of differential operators, Eq. (20') is not an ordinary differential equation in general, because $(P+G)$ is pseudo-differential.

(III) The strictly homogeneous symbols associated to $(P+G)$ and $\mathcal{B}$ have limits for $|\zeta|\to 0$ in the respective leading symbol norms, with the limiting symbol restricted to the boundary which avoids the values $\lambda \notin \mathcal{C}-\mathbf{R}_+$ for all $\{\hat{x}\}$.

Condition (III) requires a last effort for a proper understanding. Given a pseudo-differential operator of order d with leading symbol $p^0(x,\xi)$, the associated strictly homogeneous symbol is defined by[11]

$$p^h(x,\xi)\equiv |\xi|^d p^0\left(x,\frac{\xi}{|\xi|}\right) \quad \text{for} \ \ \xi\neq 0. \tag{37}$$

This extends to a continuous function vanishing at $\xi=0$ when $d>0$. In the presence of boundaries, the boundary-value problem $((P+G),\mathcal{B})$ has a strictly homogeneous symbol on the boundary equal to (some indices are omitted for simplicity)

$$\begin{pmatrix} p^h\left(\{\hat{x}\},r=0,\{\zeta\},-i\frac{\partial}{\partial r}\right)+g^h\left(\{\hat{x}\},\{\zeta\},-i\frac{\partial}{\partial r}\right)-\lambda \\ b^h\left(\{\hat{x}\},\{\zeta\},-i\frac{\partial}{\partial r}\right)\end{pmatrix},$$

where p^h,g^h and b^h are the strictly homogeneous symbols of P,G and $\mathcal{B}$ respectively, obtained from the corresponding leading symbols p^0,g^0 and b^0 via equations analogous to (37), after taking into account the split (19), and upon replacing ω by $-i\frac{\partial}{\partial r}$. The limiting symbol restricted to the boundary (also called limiting λ-dependent boundary symbol operator) and mentioned in condition III reads therefore[11]

$$a^h\left(\{\hat{x}\}, r=0, \zeta=0, -i\frac{\partial}{\partial r}\right)$$
$$= \begin{pmatrix} p^h\left(\{\hat{x}\}, r=0, \zeta=0, -i\frac{\partial}{\partial r}\right) + g^h\left(\{\hat{x}\}, \zeta=0, -i\frac{\partial}{\partial r}\right) - \lambda \\ b^h\left(\{\hat{x}\}, \zeta=0, -i\frac{\partial}{\partial r}\right) \end{pmatrix}, \tag{38}$$

where the singularity at $\xi = 0$ of the leading symbol in absence of boundaries is replaced by the singularity at $\zeta = 0$ of the leading symbols of P, G and $\mathcal{B}$ when a boundary occurs.

To figure out what sort of modifications can be introduced by the scheme just outlined, two relevant particular cases are studied hereafter.

(i) If $\widetilde{\Lambda}$ is a pseudo-differential operator of order 1, the leading symbol of the boundary operator (10) can be cast in the form (cf. (23))

$$\sigma_L(\mathcal{B}) = \begin{pmatrix} \Pi & 0 \\ i(T+\widetilde{T}) & I - \Pi \end{pmatrix}, \tag{39}$$

where $T \equiv \Gamma^j \zeta_j$ and $\widetilde{T}$ results from the occurrence of $\widetilde{\Lambda}$. The sufficient condition for finding solutions of Eq. (21') for all ψ' reads now

$$(T+\widetilde{T})^2 + |\zeta|^2 I > 0 \;\; \forall \zeta \neq 0, \tag{40}$$

because one can simply replace T with $T+\widetilde{T}$ in the analysis of Ref. 6, if Eq. (39) holds. Thus, if $\widetilde{\Lambda}$ is chosen in such a way that

$$(T+\widetilde{T})^2(\{\hat{x}\}, \{\zeta\}) > 0 \;\; \forall \zeta \neq 0, \tag{41}$$

Eq. (21') can always be solved with arbitrary $\psi'(\varphi)$. The condition (41) can be made explicit after re-writing the DeWitt supermetric (6) in the more general form

$$E^{abcd} \equiv \frac{1}{2}\Big(g^{ac}g^{bd} + g^{ad}g^{bc}\Big) + \alpha g^{ab}g^{cd}. \tag{42}$$

Thus, on defining (with $e_a^{\ i}$ being a local tangent frame on ∂M)

$$\zeta_a \equiv e_a^{\ j} \, \zeta_j, \tag{43}$$

and introducing the nilpotent matrices[6]

$$(p_1)_{ab}^{\ \ cd} \equiv n_a n_b \zeta^{(c} \, n^{d)}, \tag{44}$$

$$(p_2)_{ab}^{\ \ cd} \equiv n_{(a} \, \zeta_{b)} n^c n^d, \tag{45}$$

the work in Ref. 6 finds the useful formula

$$T = -\frac{1}{(1+\alpha)} p_1 + p_2, \tag{46}$$

and this should be inserted into (41) to restrict the kernel of $\widetilde{\Lambda}$, whose leading symbol is equal to $i\widetilde{T}$. The resulting restriction on α should be made compatible with the values of α for which the ellipticity condition (30) is fulfilled in the interior of M. From this point of view, one has definitely more choice than in the case of the local boundary operator for an operator of Laplace type on metric perturbations,[7] because the values of α for which the condition

$$T^2 + |\zeta|^2 I > 0 \ \ \forall \zeta \neq 0 \tag{47}$$

holds (cf. (40)) are incompatible with the occurrence of an operator of Laplace type on metric perturbations.[6]

(ii) If $\widetilde{\Lambda}$ is a pseudo-differential operator of order $d > 1$ (but not necessarily integer), the leading symbol of the boundary operator (10) can be expressed in the form

$$\sigma_L(\mathcal{B}) = \begin{pmatrix} \Pi & 0 \\ \widehat{T} & I - \Pi \end{pmatrix}. \tag{48}$$

The sufficient condition for finding solutions of Eq. (21') reads instead

$$-\widehat{T}^2 + |\zeta|^2 I > 0 \ \ \forall \zeta \neq 0. \tag{49}$$

It is therefore sufficient to choose $\widetilde{\Lambda}$ in such a way that

$$\widehat{T}^2 < 0 \ \ \forall \zeta \neq 0. \tag{50}$$

Indeed, non-local boundary conditions arise already in simpler problems, i.e. the quantum theory of a free particle subject to non-local boundary data on a circle.[13] One then finds two families of eigenfunctions of the Hamiltonian: *surface states* which decrease exponentially as one moves away from the boundary, and *bulk states* which remain instead smooth and non-vanishing. The generalization to an Abelian gauge theory such as Maxwell theory can fulfill non-locality, ellipticity and complete gauge invariance of boundary conditions providing one learns to work with pseudo-differential operators in one-loop quantum theory.[14] On the other hand, in the application to quantum gravity, since the boundary operator acquires new kernels responsible for the pseudo-differential nature of the boundary-value problem, one might hope to be able to recover a good elliptic theory under a wider variety of conditions. This has been shown in detail by our work, where the inequalities (30) and (34) express ellipticity in the interior of M, and an approach to the definition of strong ellipticity of the boundary-value problem has been proposed (see conditions (I), (II), (III) and the particular cases (i) and (ii) therein). It therefore turns out that, if the pseudo-differential operator $(P + G)$ and the pseudo-differential boundary operator $\mathcal{B}$ are so chosen that all the above conditions hold, invariance under infinitesimal diffeomorphisms (physics) and strong ellipticity (mathematics) can be achieved in Euclidean quantum gravity.

It would be now very interesting to prove that, by virtue of the pseudo-differential nature of $\mathcal{B}$ in (10), the quantum state of the universe in one-loop semiclassical theory can be made of surface-state type.[13] This would describe a wave function of the universe with exponential decay away from the boundary, which might provide a novel description of quantum physics at the Planck length. It therefore seems that by insisting on path-integral quantization, strong ellipticity of the Euclidean theory and invariance principles, new deep perspectives are in sight. These are in turn closer to what we may hope to test, i.e. the one-loop semiclassical approximation in quantum gravity. In the seventies, such calculations could provide a guiding principle for selecting couplings of matter fields to gravity in a unified field theory. Now they can lead instead to a deeper understanding of the interplay between non-local formulations,[15–17] elliptic theory, gauge-invariant quantization and a quantum theory of the very early universe.[9]

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